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Linear-Quadratic-Gaussian Controllers with Specified Parameter Robustness

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Despite the attractions of linear-quadratic-Gaussian (LQG) controllers, they have had limited acceptance in practice due to lack of knowledge about their robustness to uncertainties in the plant model. A suitable measure of robustness, like the gain margin or the phase margin of classical controllers, was lacking. A new robustness measure, based on the expected parameter deviations from their nominal values, is proposed here and it is used to design robust LQG controllers using nonlinear programming software.

Introduction

DYNAMIC plant models used for synthesizing feedback controllers always have some degree of uncertainty, and real plants change with age, environment, and different standards of production and maintenance. Therefore, a controller should be designed to perform satisfactorily over the anticipated range of plant parameter variations. Such a controller is said to be parameter robust. However, robustness trades with performance and so it is of interest to find the controller with best performance for a specified plant-parameter robustness; that is the subject of this paper.

Many work-years of effort go into creating controller software for an aircraft or spacecraft. Some of it is spent in making sure that the closed-loop system gives a satisfactory response for the worst possible combinations of parameter deviations. The method presented here quickly finds the worst combinations of parameter deviations and designs the best controller for handling them.

Parameter Robustness Measure

Let $\boldsymbol{p}$ be a vector of uncertain plant parameters, which has been normalized so that one unit of deviation of each element is equally likely. A parameter deviation space $P(\sigma)$ is defined as a box whose center is at $\boldsymbol{p}_{\mathrm{nom}} =$ nominal value of $\boldsymbol{p}$ and whose half-side is σ. (Figure 1 shows the box for the case with two uncertain parameters.) Controllers are synthesized using nonlinear programming software to minimize the maximum quadratic performance index J over $P(\sigma_d)$ with respect to the controller parameters, where σ_d is specified. The resulting minimax controllers often have equal maxima at several corners of P.

The parameter margin σ_p is defined as the size of the box within which the closed-loop system is stable, i.e., where $J \leq \infty$; clearly $\sigma_p > \sigma_d$.

P is the same parameter space used by Doyle[1] in his pioneering paper on robustness. This paper builds on his work and the work of

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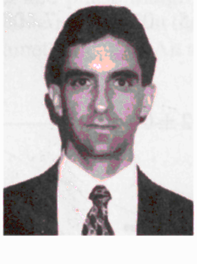

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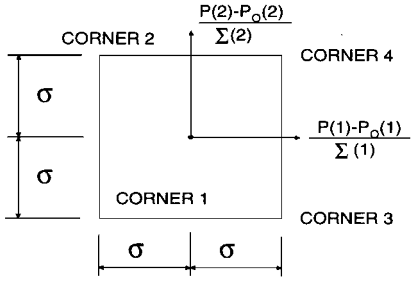

Fig. 1 Parameter space for the case of two plant parameters.

many others, but particularly on the work of Ly,[2,3] El Ghaoui et al.,[4] El Ghaoui and Bryson,[5] Mills,[6] Mills and Bryson,[7] and Fischer and Psiaki.[8]

Optimal Robust Controllers

Suppose the designer wishes the controlled performance to be satisfactory inside a box of size σ_d in the normalized parameter space. One way of doing this is to choose the controller parameter vector $\boldsymbol{k}$ to minimize and $\boldsymbol{p}$ to maximize a quadratic performance index J, i.e.,

$$\min_{k} \max_{p} J(\boldsymbol{k}, \boldsymbol{p}) \tag{1}$$

over $P(\sigma_d)$. This corresponds to finding the worst combination of parameter deviations in the box and the best controller for that combination. The idea and the compact $(\boldsymbol{k}, \boldsymbol{p})$ nomenclature were suggested by Ref. 8.

The minimum and maximum operations must be done simultaneously because the best controller depends on the worst parameter variations and vice versa. This is a complicated design problem; it is considerably simplified if only a finite number of deviated plants are considered, i.e., if we use a multiplant model as suggested by Ref. 2. The multiple plants selected here are the plants corresponding to the corners of $P(\sigma_d)$. This corresponds to a uniform probability density for the elements of $\boldsymbol{p}$ in the box $P(\sigma_d)$. This box has 2^N corners; Fig. 1 shows the case where $N = 2$ where there are $2^2 = 4$ corners.

Multiple Minimax Points

For a system with many uncertain parameters there may be several bad combinations of parameter deviations, i.e., several bad corners of P. Synthesizing a controller that decreases J at the worst corner may cause J to increase at some of the other bad corners. In fact, after several gradient iterations, J at some other bad corner may become equal to or larger than the maximum at the original design corner; now the designer must decrease the maximum at the original design corner with the constraint that J at this second corner stays less than or equal to the value of J at the original design corner. In effect the controller parameters are being found to minimize two equal maxima simultaneously (as suggested in Ref. 8). This idea can obviously be extended to the case where more than two bad corners have equal maxima. Professional nonlinear programming software is available that finds a set of parameters to minimize the maximum of several functions of these parameters, e.g., the MINIMAX command in the MATLAB Optimization Toolbox. The several functions chosen here are the values of J at a subset of corners, which are determined by evaluating J at all of the corners and choosing those corners (3–6 in our example) having the highest values of J. The authors' experience is that the true maxima of J usually occur on the edges of the box, but the corner approximation appears to be accurate enough for controller design and is considerably more efficient than looking for the exact location of the maximum. Furthermore El Ghaoui and Bryson[5] have shown that the maxima always occur at the corners for conservative dynamic systems (systems with no damping). Thus, it seems plausible that the maxima will occur close to the corners for systems with lightly damped modes, and these are precisely the systems where robustness must be added when designing linear-quadratic-Gaussian (LQG) controllers.

When the design is complete, we check for possible interior maxima by evaluating the maxima of J at the corners of several nested interior boxes; this amounts to a multidimensional discretization of the interior of the box.

A particular corner of a box is specified by giving σ and the signs of the deviations of each element of $\boldsymbol{p}$. We adopted the convention of numbering the corners by replacing the $(-)$ signs by zeros and the $(+)$ signs by ones and converting the corresponding binary number to a decimal number. For example, if there are six plant parameters, then the corner where the perturbations are $[+ + - + - +]$ converts to the binary number 110101, which is $1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 53$. Because MATLAB does not handle 0 as an index number, we add 1 to make this corner number 54. This is a convenience for the control designer because then the changes in the six parameters can easily be found given the corner number.

Example: Helicopter Position Controller

To explain the synthesis method, we consider the design of LQG controllers to keep a helicopter hovering near a point on the ground in the presence of a random wind disturbance $u_w(t)$ (Fig. 2). Performance is measured by the expected value of the quadratic performance index

$$J = y^2 + \delta^2 \tag{2}$$

where y is the deviation of the helicopter from the desired hover point in feet and δ is the deflection of the longitudinal cyclic stick (the control) in deci-inches. Here, δ is proportional to the tilt of the rotor tip-path plane relative to the fuselage (Fig. 2). By choosing this J we indicate that we are willing to use 1 deci-in. of stick deflection when there is 1 ft of position deviation.

We used a fourth-order model with state vector $\boldsymbol{x} = [u \; q \; \theta \; y]^T$ where u is the horizontal velocity of the helicopter relative to the ground in feet per second, $q = \dot{\theta}$ in centi-radians per second, and θ is the pitch angle of the fuselage in centi-radians. The disturbance is u_w equal to horizontal wind velocity in feet per second. The model has six uncertain parameters $\boldsymbol{p} = [p(1) \cdots p(6)]$. The first four are the aerodynamic stability derivatives X_u, X_q, M_u, and M_q, and the last two are the aerodynamic control derivatives X_δ and M_δ. Because $[\boldsymbol{p}]$ is a six vector, there are $2^6 = 64$ corners of the box in the normalized $\boldsymbol{p}$ space.

The model is $\dot{x} = Ax + B\delta + B_w u_w$, where

$$A = \begin{bmatrix} p(1) & p(2) & -0.322 & 0 \\ p(3) & p(4) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} p(5) \\ p(6) \\ 0 \\ 0 \end{bmatrix}, \qquad B_w = \begin{bmatrix} -p(1) \\ -p(3) \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

The nominal values of the uncertain parameters are[9]

$$p_0 = [-0.0257 \quad 0.013 \quad 1.26 \quad -1.765 \quad 0.086 \quad -7.408] \tag{4}$$

The open-loop system has eigenvalues

$$\mathrm{ev} = [0 \quad -1.8891 \quad 0.0492 \pm 0.4608j] \tag{5}$$

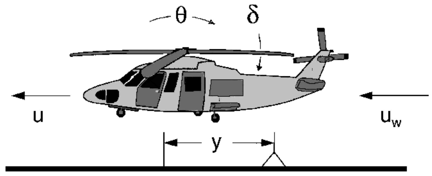

Fig. 2 Example of helicopter hovering over a fixed point with a wind disturbance $u_w(t)$.

which correspond, respectively, to a position mode, a damped pitching mode, and a slightly unstable phugoid mode.

We normalized each element of $\boldsymbol{p}$ by 15% of the absolute value of the corresponding element of p_0. This is a guess; better normalizations should be obtained from the experts who constructed the aerodynamic model. Thus, when $\sigma = 1/0.15 = 6.67$, one corner of the box has zero control derivatives; clearly all possible closed-loop systems have parameter margins $\sigma_p \leq 6.67$.

We considered three different sets of sensors: 1) all states measured with negligible error, 2) two sensors (position y and pitch angle θ) each with additive white noise, and 3) one sensor (position y) with additive white noise. Clearly we will obtain the best controllers (smallest values of J) with set 1, the next best with set 2, and the poorest with set 3.

We first synthesized the standard linear quadratic regulator (LQR) or LQG controller and determined its parameter margin. We then synthesized an LQR or LQG controller with a higher parameter margin. We chose controllers having the same order as the plant model, but we could have chosen them to have higher or lower order. Lower-order controllers are simpler to implement but have poorer performance, whereas higher-order controllers are more difficult to implement and have better performance. We chose the design parameter vector $\boldsymbol{k}$ to be the state feedback gain vector $\boldsymbol{K}$ plus the elements of the estimator gain matrix L. However, any nonredundant parametrization of the controller can be used.

Controllers with All States Measured with Negligible Error

For this case the standard LQG controller has a parameter margin of 3.95 (Fig. 3). Figure 3 is a plot of $1/\sqrt{(J_w)}$ vs σ, obtained by calculating the quadratic performance index J at all 64 corners of six boxes with sides $\sigma = 1, 2, 3, 3.5, 3.75$, and 3.95, and taking the maximum, i.e., the worst, value J_w for each value of σ. This takes only half a minute with a 100-MHz Pentium personal computer using the MATLAB Control System Toolbox.[10] This suggested that the maximum values of J do occur near the corners of the $\sigma = 3.95$ box. The critical corner of the plant parameter space for $\sigma = 1, 2, 3$, and 3.5 is number 30, where the plant parameter perturbations are $[-,+,+,+,-,+]$, whereas at $\sigma = 3.75$ and 3.95 the critical corner is number 62, where the plant parameter perturbations are $[+,+,+,+,-,+]$. Both of these corners have decreased magnitudes of the control derivatives X_δ and M_δ.

We then designed a more robust state feedback controller with a design parameter margin $\sigma_d = 6$, using the MINIMAX command of the MATLAB Optimization Toolbox[11]; see Appendices A and B for details. Figure 4 shows that there are two equal maxima at corners 30 and 64 (corner 64 has all positive plant parameter perturbations $[+,+,+,+,+,+]$). The parameter margin is $\sigma_p = 6.2$.

With this controller, we then calculated the quadratic performance index J at all 64 corners of five interior and two exterior boxes, taking the maximum, i.e., the worst, value J_w for each σ [the other two points are at the nominal ($\sigma = 0$) and at the design box $\sigma = 6$]. Figure 3 shows that this controller is, indeed, more robust than the standard LQG design; however, this robustness is achieved by having poorer performance when the parameters are near their nominal values.

For comparison with classical robustness measures, we calculated the gain and phase margins for both controllers with the nominal plant. Table 1 shows that these classical measures correlate well with the parameter margin. An advantage of the parameter margin is that it is tailored to the designer's physical knowledge of the plant parameter uncertainties; a disadvantage is that it is more complicated to compute.

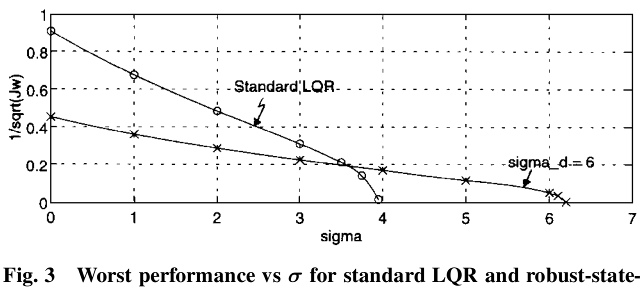

Fig. 3 Worst performance vs σ for standard LQR and robust-state-feedback controllers.

Table 1 All states known

Controller	Plant	Gain margin	Phase margin, deg	Parameter margin
LQG	Nominal	∞	60.1	3.95
Robust	Nominal	∞	91.8	6.20

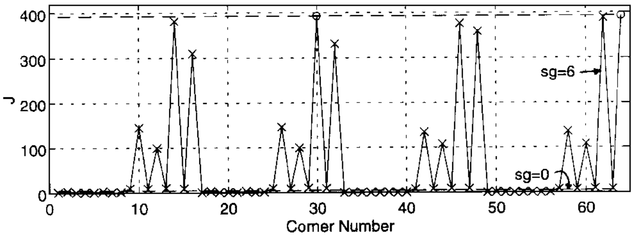

Fig. 4 Performance index J vs corner number for robust state feedback controller with σ_d = 6.

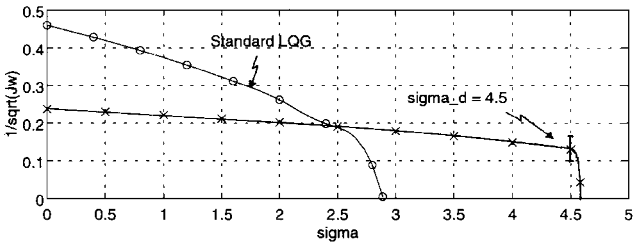

Fig. 5 Worst performance vs σ for standard LQG and robust two-sensor controllers.

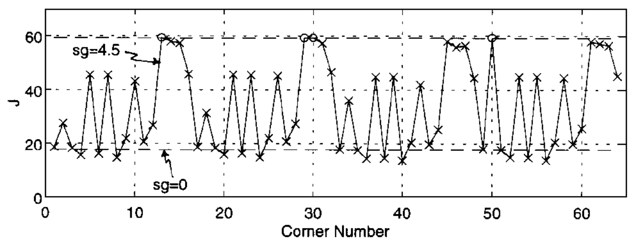

Fig. 6 Performance index J vs corner number for robust two-sensor controller with σ_d = 4.5.

Controllers with Two Noisy Sensors

For this case the standard LQG controller has a parameter margin of 2.89 and, as expected, poorer perfomance than the state feedback controllers (Fig. 5). The critical corner of the plant parameter space was 30 for $\sigma \leq 2$ and 14 for $\sigma > 2$.

We then designed a controller with a design parameter margin of $\sigma_d = 4.5$. Figure 6 shows that, to the resolution of the figure, there are four equal maxima at corners 13, 29, 30, and 50.

With this controller we then calculated the quadratic performance index J at all 64 corners of eight interior boxes and one exterior box, taking the maximum, i.e., the worst, value J_w for each σ. Figure 5 shows again that the robust controller has poorer performance than the LQG controller when the parameters are near their nominal values.

For comparison with classical robustness measures, we calculated the gain and phase margins for both controllers with the nominal plant and for the robust controller with one of the worst plants (at corner 50). Table 2 shows that again the classical robustness measures correlate well with the parameter-margin measure.

Controllers with One Noisy Sensor

For this case the standard LQG controller has a parameter margin of only 0.66 and, as expected, has poorer performance than the two-sensor controllers (Fig. 7). The critical corners of the plant parameter

Table 2 Noisy y and θ sensed

Controller	Plant	Gain margins	Phase margin, deg	Parameter margin
LQG	Nominal	3.40, 0.42	27.1	2.89
Robust	Nominal	2.97, 0.13	39.7	4.58
Robust	Worst	11.6, 0.94	4.1	0.08

Table 3 Only noisy y sensed

Controller	Plant	Gain margins	Phase margin, deg	Parameter margin
LQG	Nominal	1.30, 0.49	12.6	0.66
Robust	Nominal	1.71, 0.32	26.8	1.69
Robust	Worst	1.08, 0.41	4.1	0.19

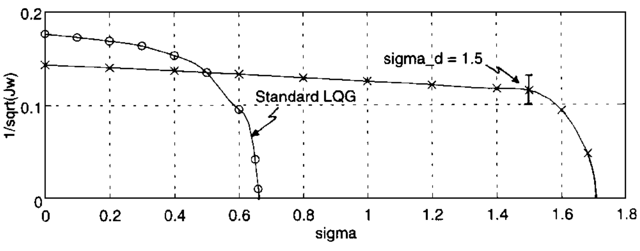

Fig. 7 Worst performance vs σ for standard LQG and robust one-sensor controllers.

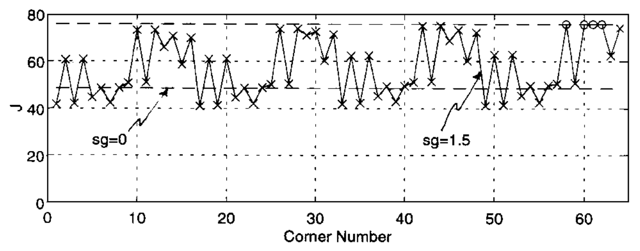

Fig. 8 Performance index J vs corner number for robust one-sensor controller with σ_d = 1.5.

space were numbers 62 (which was also a critical corner in the earlier state feedback case) and 61, where the plant parameter variations are $[+, +, +, +, -, -]$.

We then designed a more robust controller with a design parameter margin of $\sigma_d = 1.5$. Figure 8 shows the performance index J at all 64 corners of the $\sigma = 1.5$ box; to the resolution of the figure, there are four equal maxima at corners 58, 60, 61, and 62 (corner 58 has plant parameter perturbations $[+, +, +, -, -, +]$).

With this controller we then calculated J at all 64 corners of seven interior boxes and two exterior boxes, taking the maximum, i.e., the worst, value J_w for each σ. Figure 7 shows that this design is more robust than the standard LQG design, but again, as expected, it has poorer performance when the parameters are near their nominal values.

For comparison with classical robustness measures, we calculated the gain and phase margins for both controllers with the nominal plant and for the robust controller with one of the worst plants (at corner 61). Table 3 shows again that the classical robustness measures correlate well with the parameter-margin measure.

Figure 9 shows a Nyquist plot for the standard LQG and the robust controllers with the nominal plant. An interesting feature is that the vector margin is nearly constant over a large range, i.e., the M circle about $s = -1$ almost coincides with the robust controller plot for ± 60 deg about the real axis.

As a further check we fit a six-dimensional quadric surface to the computed points at the corners of the box and at the origin (using the MATLAB Optimization Toolbox command LEASTSQ). The fit was very good and the maximum of this continuous $J(\Delta p)$ occurred on an edge of the box between two of the critical corners.

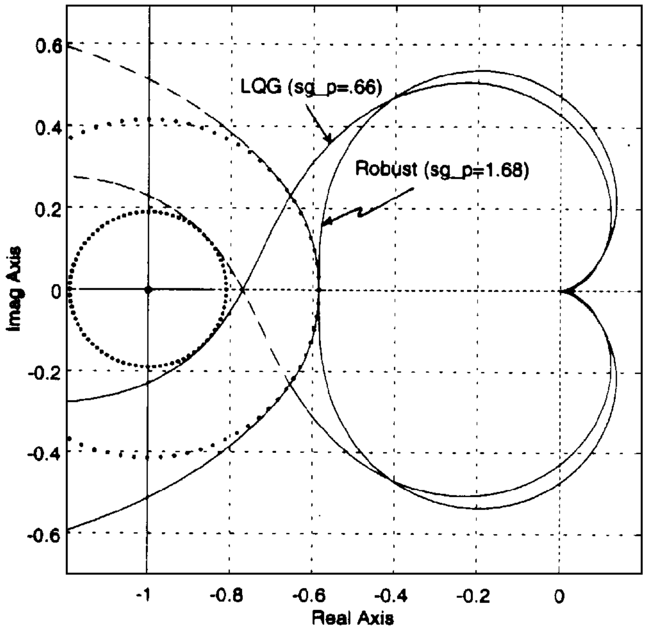

Fig. 9 Nyquist plots for the standard and robust one-sensor controllers with the nominal plant.

Comparisons with Other Robust Design Methods

Ly[2] gave one of the first robust control algorithms that uses multiple models. The controller was designed by minimizing a weighted sum of quadratic performance indices (QPIs), each one for a plant with different parameters; the plants and the weighting factors are chosen by the designer. In the minimax method used here, the plants and the weighting factors are chosen inside the code; the plants chosen are the worst plants, and the weighting factors are chosen to make the maxima equal and to be zero on all submaximum points.

El Ghaoui et al.[4] and El Ghaoui and Bryson[5] also used multiple models with a linear quadratic (LQ) performance index and worst-plant parameter variations, but they used worst initial conditions of magnitude unity as the disturbance instead of the random process and sensor disturbances used here. This is a more rational choice for some problems, where the process disturbances are negligible, e.g., in flexible spacecraft; this method also involves solving a Lyapunov equation for each model.

Mills[6] also used multiple models with an LQG performance index, making the ith model the worst case for $\sigma = \sigma_i$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_d$ and σ_d is the radius of a hypersphere (instead of a hypercube or box) in the specified parameter variation space. He chose the weighting factors to be 0.99 on the nominal and 0.01 on the plant corresponding to the parameter margin, which shapes the worst J vs σ curve to give better performance near $\sigma = 0$ and poorer performance near $\sigma = \sigma_d$. This method[7] was used to check the example here; in all cases the parameter-margin (PM) hypersphere was between the hypersphere that circumscribes and the hypersphere that is tangent to the PM hypercube or box.

Fischer and Psiaki[8] suggested minimaxing over the allowable parameter variation space, and this is the method we have described in this paper. The designer does not have to choose weighting factors on the different models because the minimax operation involves selecting the weights to make the multiple maxima equal. The MATLAB Optimization Toolbox MINIMAX code[11] greatly facilitates these computations because it does not require analytical gradients, only expressions for the performance index and the constraints. Gradients are calculated numerically inside the code, and second derivatives (the Hessian matrix) are estimated during the iterative computations. Without such software one has to generate analytical expressions for the gradients of the performance index and the constraints; however, once this is done, the computations proceed more rapidly.

Balas et al.[12] have developed a method they call μ synthesis for robustifying H_∞ controllers to parameter variations. It treats the plant parameters $\boldsymbol{p}$ as complex numbers (making the method conservative) and scales the Δp feedbacks to approximately maximize the performance index with respect to Δp. It requires introducing high-pass filters on the parameter variation feedbacks inasmuch as unconstrained H_∞ controllers have infinite bandwidths; this results in controllers of substantially higher order than the plant model. Our (very limited) experience with such controllers[6] indicates that they do not differ significantly from the robust LQG controllers described here; the latter are simpler to synthesize because they do not require the ad hoc filters to limit controller bandwidth. In Ref. 13, the authors give a revised version of μ synthesis with real parameter variations but it is restricted to the case of full state feedback.

Summary and Conclusions

Using an example with six uncertain plant parameters, we have described a method for synthesizing LQG controllers that minimizes the maximum quadratic performance index at the corners of the specified plant parameter space. The method uses a minimax nonlinear programming code and a code that solves Lyapunov equations. The resulting controllers have equal maxima of the quadratic performance index at several corners of the specified plant parameter space.

For the helicopter controller example, the PM over standard LQR or LQG was increased by factors of 1.57, 1.58, and 2.56 for the state feedback and two-sensor and one-sensor cases, respectively. This increased the stable volume in the parameter space by these factors to the sixth power, or 15, 16, and 280.

Performance of the plant with nominal parameters always degrades as robustness to parameter variations is increased.

Appendix A: Closed-Loop Dynamic System

The closed-loop system is a combination of the plant and controller dynamics that is two-way coupled inasmuch as the sensed outputs from the plant are inputs to the controller and the control signals (outputs of the controller) are inputs to the plant. The augmented system is

$$\dot{x}_a = A_a x_a + B_{wa} w_a \tag{A1}$$

where

$$x_a = \begin{bmatrix} x \\ x_c \end{bmatrix}, \qquad w_a = \begin{bmatrix} w \\ v \end{bmatrix}, \qquad u = -K x_c \tag{A2}$$

$$A_a = \begin{bmatrix} A & -BK \\ LC_s & A_0 - B_0 K - LC_s \end{bmatrix}, \qquad B_{wa} = \begin{bmatrix} B_w & 0 \\ 0 & L \end{bmatrix} \tag{A3}$$

$$Q_a = \begin{bmatrix} C^T Q C & 0 \\ 0 & K^T R K \end{bmatrix}, \qquad W_a = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \tag{A4}$$

and where $A = A(\boldsymbol{p})$, $B = B(\boldsymbol{p})$, $B_w = B_w(\boldsymbol{p})$, whereas $A_0 = A(\boldsymbol{p}_0)$, $B_0 = B(\boldsymbol{p}_0)$ because the estimator only knows the nominal values of the plant parameters.

The performance index is obtained by first solving the Lyapunov equation

$$A_a X_a + X_a A_a^T + B_{wa} W_a B_{wa}^T = 0 \tag{A5}$$

and then calculating

$$J = \mathrm{tr}(X_a Q_a) \tag{A6}$$

For the state feedback case there is no estimator (no v, V, and L), and the performance index is obtained by first solving the Lyapunov equation

$$(A - BK)X + X(A - BK)^T + B_w W B_w^T = 0 \tag{A7}$$

and then calculating

$$J = \mathrm{tr}[X(C^T Q C + K^T R K)] = 0 \tag{A8}$$

Appendix B: Helicopter Hover Example

The numerical solutions were obtained using the MINIMAX command of the MATLAB Optimization Toolbox, which called the function files listed subsequently. This command does not require an analytical expression for the gradients. See Appendix C for an example run.

```
function [f,g]=oh6_sfb(k,Nc,sg)
% OH6 with 6 uncertain plant parameters, STATE FEEDBACK with
% noise-free measurements; x=[u,q,th,y]'; k is the state
% feedback gain row vector; Nc is a row vector of critical
% corner numbers; sg is the half-side of the p-space box.
%
% Nominal plant data and performance index weights:
%
po=[-.0257 .013 1.26 -1.765 .086 -7.408];
C=[0 0 0 1];  Q=1;   R=1;  W=18;
Qa=C'*Q*C+k'*R*k;  Sg=diag(.15*abs(po));
%
% Performance index at corners specified by Nc:
%
nc=length(Nc);
for i=1:nc;  dp=nc2dp(Nc(i));  p=po+sg*dp*Sg;
  A=[p(1) p(2) -.322 0; p(3) p(4) 0 0; 0 1 0 0; 1 0 0 0];
  B=[p(5) p(6) 0 0]';   Bw=[-p(1) -p(3) 0 0]';
  X=lyap(A-B*k,Bw*W*Bw');  f(i)=sum(diag(X*Qa));
end;
g=[];

function [f,g]=oh6_2sen(k,Nc,sg)
% OH6 with 6 uncertain plant parameters, 4th order compensator,
% 2 SENSORS, ys=[y, theta]'; x=[u,q,th,y]'; the controller
% parameter vector is k=[K L(:,1)' L(:,2)'];  Nc = row vector
% of critical corner numbers;  sg is the half-side of the
% p-space box.
%
% Nominal plant data and perf. index weights:
%
```

```
po=[-.0257 .013 1.26 -1.765 .086 -7.408];
Ao=[po(1) po(2) -.322 0;po(3) po(4) 0 0;0 1 0 0;1 0 0 0];
Bo=[po(5) po(6) 0 0]';
C=[0 0 0 1];             Q=1;  R=1;
Cs=[0 0 0 1; 0 0 1 0];   Wa=diag([18 .4 .4]);
Sg=diag(.15*abs(po));
%
% Performance index at corners specified by Nc:
%
K=k([1:4]);  L=[k([5:8]);  k([9:12])]';  nc=length(Nc);
Qa=[C'*Q*C zeros(4); zeros(4) K'*R*K];
for i=1:nc,
  dp=nc2dp(Nc(i));  p=po+sg*dp*Sg;
  A=[p(1) p(2) -.322 0;p(3) p(4) 0 0;0 1 0 0;1 0 0 0];
  B=[p(5) p(6) 0 0]';  Bw=[-p(1) -p(3) 0 0]';
  Aa=[A -B*K; L*Cs Ao-Bo*K-L*Cs];
  Bwa=[Bw zeros(4,2); zeros(4,1) L];
  Xa=lyap(Aa,Bwa*Wa*Bwa');  f(i)=sum(diag(Xa*Qa));
end;
g=[];

function [f,g]=oh6_1sen(k,Nc,sg)
% OH6 helicopter with 6 uncertain plant parameters, 4th order
% compensator, 1 SENSOR; x = [u,q,th,y]'; the controller
% parameter vector is k = [K L']; Nc = row vector of
% critical corner numbers;  sg is the half-side of the
% p-space box.
%
% Nominal plant and performance index weights:
%
po=[-.0257 .013 1.26 -1.765 .086 -7.408];
Ao=[po(1) po(2) -.322 0;po(3) po(4) 0 0;0 1 0 0;1 0 0 0];
Bo=[po(5) po(6) 0 0]';
C=[0 0 0 1];  Cs=C;  Q=1;  R=1;  Wa=diag([18 .4]);
Sg=diag(.15*abs(po));
%
% Performance index at corners specified by Nc:
%
K=k([1:4]);   L=k([5:8])';   nc=length(Nc);
Qa=[C'*Q*C zeros(4); zeros(4) K'*R*K];
for i=1:nc, dp=nc2dp(Nc(i)); p=po+sg*dp*Sg;
  A=[p(1) p(2) -.322 0; p(3) p(4) 0 0; 0 1 0 0; 1 0 0 0];
  B=[p(5) p(6) 0 0]';   Bw=[-p(1) -p(3) 0 0]';
  Aa=[A -B*K; L*Cs Ao-Bo*K-L*Cs];
  Bwa=[Bw zeros(4,1); zeros(4,1) L];
  Xa=lyap(Aa,Bwa*Wa*Bwa');     f(i)=sum(diag(Xa*Qa));
end;
g=[];
```

All three of the preceding codes use the command nc2dp, which converts the decimal corner number nc to the six-column row vector $\boldsymbol{dp}$, which indicates in which direction each of the plant parameters changes (either $+1$ or -1). This is essentially a conversion of a decimal number to the corresponding binary number where the zeros in the binary number are replaced by -1s. The code is given as follows:

```
function dp=nc2dp(nc)
% Finds dp given a decimal corner number nc.
%
for j=1:6
  if nc-1<2^(6-j), dp(j)=-1;
  else             dp(j)=+1; nc=nc-2^(6-j);
  end
end
```

The results for the standard LQ and the robust controllers are listed in the edited MATLAB diary as follows:

```
% Optimal Robust compensators for OH6 helicopter
%
% State feedback designs; sigma_d unspecified and sigma_d=6:
k0 =  1.9890   -0.2560   -0.7589    1.0000
sg0 =    0    1.00    2.00    3.00    3.50    3.75    3.95
Nc0 =    1     62     62     61     61     61     61
y0 = .9056  .6750  .4846  .3095  .2117  .1400  .0128
k6 =  4.2761   -2.2876   -2.8748    0.4504
```

```
sg6 =  0   1.0   2.0   3.0   4.0   5.0   6.0   6.1   6.2
Nc6 =  1  14  14  14  14  14  (30,64)  64  64
y6 = .4527  .3606  .2868  .2246  .1692  .1156  .0505
     .0353  .0029

% 2 sensor designs: sigma unspecified and sigma_d=4.5:
k0 = 1.9890 -.2560  -.7589  1.0000  .3188  .1373 -.0810
      .7943 -.4180  3.5737  2.6722 -.0810
sg0 =  0  0.40  0.80  1.20  1.60  2.00  2.40  2.80  2.89
Nc0 = 1   30   30   30   30   30   14   14   14
y0 =.4597  .4279  .3930  .3547  .3117  .2618  .1982  .0880  .0051
k45 = 3.4822 -0.2072 -2.3604  3.0467  0.1785  1.8587 -0.1343
      0.3844  0.5890 38.1416  7.8574  2.2859
sg45 = 0  .50  1.00  1.50   2.00  2.50  3.00  3.50  4.00  4.50  4.58
Nc45 = 1  29  29  29  29  29  29  29  29  (13,29,30,50)  34
y45 = .2376  .2300  .2216  .2125  .2025  .1915  .1792
      .1653  .1491  .1299  .0429

% 1 sensor designs: sigma_d unspecified and sigma_d=1.5:
k0 = 1.9890 -.2560  -.7589  1.0000  2.3172    .7218  -3.661 2.1528
sg0 = 0  .100  .200  .300  .400  .500  .600  .650  .659
Nc0 = 1   62   62   61   61   61   61   61   61
y0 = .1763  .1724  .1682  .1631  .1527  .1347  .0955  .0421  .0110
k15 = 1.3829 -.8350 -1.3121   .4112  5.7019 -44.5805  20.215 -.4028
sg15 = 0  .20  .40  .60  .80  1.00  1.20   1.40  1.50  1.60  1.68
Nc15 = 1  44  44  44  60  60  60  (58,60,61,62)  61  61
y15 = .1436  .1402  .1367  .1331  .1293  .1253  .1213  .1171  .1150
      .0940  .0475
```

Appendix C: Example Run Using MINIMAX

This is an example run using the MINIMAX command of the MATLAB Optimization Toolbox, which starts with a converged solution to keep it short. We started with the LQG solution, which has $\sigma_p = 0.66$, and used $\sigma_d = 0.5$ and tight upper and lower bounds (vlb and vub); several iterations were required for convergence, as we gradually opened up the bounds. Then we moved to $\sigma = 0.7$ and repeated the procedure, moving in steps of about $\Delta\sigma_d = 0.2$ until we reached $\sigma_d = 1.5$. This is tedious; a good programmer could certainly develop a better algorithm.

```
% Edited diary of a 'minimax' computation for the 1-sensor robust
% controller with sg_d=1.5 (k=k15 already converged);
%
k15=[1.3829 -.8350 -1.3121 .4112 5.7019 -44.5805 20.2154 -.4028];
opt=[1 .0005 .01 0 0 0 0 0 0 0 0 0 0 500];        % Sets options
vlb=k15-.2*ones(1,8);                         % Sets lower bounds on k
vub=k15+.2*ones(1,8);                         % Sets upper bounds on k
Nc15d=[58 60 61 62];                         % Critical corner numbers
sg=1.5;                                                   % Sets sg_d
k15a=minimax('oh6_1sen',k15,opt,vlb,vub,[],Nc15d,sg)
f-COUNT     MAX{g}         STEP  Procedures
   10      75.6295            1   ok
   --        ---             ---
  158      75.6295            1   mod Hess ok
Optimization Converged Successfully; Active Constraints: 1  3  4
k15a=[1.3834 -.8351 -1.3121 .4113 5.7015 -44.5724 20.2112 -.4025]
%
% Iterate again with no bounds ('vlb' and 'vub' not specified):
%
k15b=minimax('oh6_1sen',k15a,opt,[],[],[],Nc15d,sg)
f-COUNT     MAX{g}         STEP  Procedures
   10      75.6295            1   ok
   --        --               -    -
  126      75.6295            1   ok
Optimization Converged Successfully; Active Constraints: 1  3  4
k15b=[1.3809 -.8345 -1.3103 .4103 5.7014 -44.5685 20.2099 -.4023]
%
% Check that J is positive at all 64 corners and that ' Nc15d'
% contains the critical corners:
%
f=oh6_1sen(k15b,1:64,1.5);   [J,l]=sort(f)
J=[40.9413  41.1303  41.2881  41.4867  41.6958  41.8645  41.9956
     ---    74.9338  74.9662  75.6081  75.6295  75.6295  75.6295
l=[17    49    19    51     1    33    23
    --   42    44    60    58    62    61
% Checks OK
```

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